Duality and self-duality in Ginzburg-Landau theory with Chern-Simons term

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(Dated: Received February 1, 2008)

We derive the exact dual theory of a lattice Ginzburg-Landau theory with an additional topological Chern-Simons (CS) term. It is shown that in the zero-temperature limit, the statistical parameter $\theta = 1/2\pi$ corresponds to a fixed point of the duality transformation. Thus we have found a nontrivial example of self-duality in three dimensions. In this scenario, the specialization of anyonic to fermionic statistics my be viewed as a phase transition.

PACS numbers: 74.20.-z, 05.10Cc, 11.25.Hf

Duality is an important tool for studying the strong-coupling regime of field theories. Most powerful is the property of self-duality, which in historic example of the two-dimensional Ising model allowed for an exact determination of the critical temperature before the Onsager solution [1]. A direct consequence of self-duality is the exact equality of the amplitudes of the leading term $A_{\pm}|T/T_c-1|$ of the specific heat above and below the critical temperature: $A_+/A_- = 1$. The same result holds for two-dimensional q-state Potts model [2] since it is also self-dual.

There are only very few examples for self-duality in dimension $d > 2$ [3]. The four-dimensional Z_2 gauge theory is self-dual determining exactly the critical temperature. In three dimensions, the Z_2 gauge theory coupled to bosonic matter fields is self-dual. A common feature of these examples is the discreteness of the symmetry group. For continuous symmetries, non-trivial models with self-duality are very hard to find.

Take for example the three-dimensional $U(1)$ theory discussed by Townsend *et al.* [4] and Deser and Jackiw [5]. It is self-dual, but noninteracting, describing a vector field $\mathbf{A}$ with a Hamiltonian [4, 6]

$$\mathcal{H}_0 = \frac{\theta^2}{2}\mathbf{A}^2 + i\frac{\theta}{2}\mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) \tag{1}$$

Its self-duality is obvious by the equation of motion $\mathbf{A} = -i(\boldsymbol{\nabla}\times\mathbf{A})/\theta$, and $\boldsymbol{\nabla}\times\mathbf{A}$ is dual to $\mathbf{A}$. It has been pointed out by Deser and Jackiw [5] that Eq. (1) is equivalent to the locally gauge-invariant model

$$\mathcal{H}_0' = \frac{1}{2}(\boldsymbol{\nabla}\times\mathbf{B})^2 + i\frac{\theta}{2}\mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{B}). \tag{2}$$

The Hamiltonian (1) is in fact the dual of the Hamiltonian (2), the latter being locally gauge-invariant, while the former is not.

In this note we want to exhibit a nontrivial self-dual model containing a topological Chern-Simons (CS) term [7] which may serve as an effective field theories for condensed matter systems. Such effective field theories have recently become useful tools for understanding a variety of new low-temperature phenomena [8]. A CS field permits us to attach flux tubes to particles in three dimensions, thereby changing continuously their statistics from bosonic to fermionic [9]. In the context of the fractional quantum Hall effect, the CS mass is associated to the phase determining the statistics of Laughlin quasiparticles [8]. A typical effective theory in this context consists of a Ginzburg-Landau (GL) the the Maxwell term $(\boldsymbol{\nabla}\times\mathbf{A})^2/2$ is exchanged by a CS-term [10]. If we want to perform renormalization studies of such a model, the Maxwell term must be included in addition to the CS term to provide the theory with a gauge-invariant cutoff [11]. This model will be be called GLCS model, and has a Hamiltonian

$$\begin{aligned}\mathcal{H}_e &= \frac{1}{2e^2}(\boldsymbol{\nabla}\times\mathbf{A})^2 + i\frac{\theta}{2}\mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) + |(\boldsymbol{\nabla}-i\mathbf{A})\psi|^2 \\ &+ r|\psi|^2 + \frac{u}{2}|\psi|^4,\end{aligned} \tag{3}$$

where ψ is a complex order field. The limiting Hamiltonian $\mathcal{H}_\infty$ gives an effective description of anyonic quasiparticles for statistical parameters $0 < \theta < 1/2\pi$. A bosonized fermion theory lies at the upper end of this interval: $\theta = 1/2\pi$. For arbitrary e and θ, the critical behavior of the above model has been studied before [12]. It exhibits continuously varying critical exponents due to the fact that the CS-term remains unrenormalized [11]. More recently, the GLCS model appeared as an effective dual theory for the spin sector of a strongly correlated electron system [13]. The spin sector is originally a fermionic theory, describing part of the effective dynamics of the $t-J$ model, which in bosonized form contains a CS-term. Since the low-energy theories associated with the $t-J$ model are typically gauge theories at infinite coupling, the corresponding bosonized theory has $\theta = 1/2\pi$ and no Maxwell term. The corresponding dual theory of the spin sector, however, contains a Maxwell term and has the form (3) [13]. This is to be expected, since a duality transformation maps a strongly coupled to a weak coupled theory.

Here we shall discuss the interesting duality properties of the Hamiltonian $\mathcal{H}_\infty$ using the approach introduced for the GL model in Refs. 15 and 16. We perform the duality

transformation *exactly* in a lattice GLCS model at $e = \infty$ and show that this generates a Maxwell term in the dual theory. The duality transformation will be shown to have a fixed point at $\theta = 1/2\pi$ and zero temperature. There the theory is self-dual. At the parameter $\theta = 1/2\pi$ the statistics of the order field becomes purely fermionic. The self duality extends over the entire critical regime of the model since the generated Maxwell is irrelevant for the renormalization group flow. Our results implies that anyons become fermions in a phase transition.

The duality transformation of $\mathcal{H}_\infty$ can be performed *exactly* if we consider the London limit in which the size of the complex order field is fixed, put the model on a lattice, and approximate it à la Villain with a high accuracy [14]. Within this philosophy, the lattice version of the Hamiltonian $\mathcal{H}_e$ has the form

$$\begin{aligned} H_e &= \sum_x \Bigg[\frac{\beta J}{2} \sum_\mu (\boldsymbol{\nabla}_\mu \varphi_x - 2\pi n_{x\mu} - A_{x\mu})^2 \\ &+ \frac{1}{2e^2} (\boldsymbol{\nabla} \times \mathbf{A}_x)^2 + i\frac{\theta}{2} \mathbf{A}_x \cdot (\boldsymbol{\nabla} \times \mathbf{A}_x) \Bigg], \end{aligned} \tag{4}$$

where the lattice derivative is $\boldsymbol{\nabla}_\mu f_x \equiv f_{x+\mu} - f_x$, and $\beta = 1/T$. The partition function is given by

$$Z = \int \left[\prod_x \frac{d\varphi_x}{2\pi} d\mathbf{A}_x \right] \sum_{\mathbf{n}_x} \exp(-H_e), \tag{5}$$

where the sum runs over all integer $n_{x\mu}$ and the domains of integration are $\varphi_x \in (-\pi, \pi)$ and $A_{x\mu} \in (-\infty, \infty)$.

Let us set $e^2 = \infty$. Following standard techniques [15, 16, 17, 18] we rewrite $(\beta J/2)(\boldsymbol{\nabla}_\mu \varphi_x - 2\pi n_{x\mu} - A_{x\mu})^2$ as $\mathbf{B}_x^2/(2\beta J) + i\mathbf{B}_{x\mu}(\boldsymbol{\nabla}_\mu \varphi_x - 2\pi n_{x\mu} - A_{x\mu})$ and apply Poisson's formula to convert the integrals over $\mathbf{B}_x$ into a sum over integer variables $\mathbf{b}_x$, we obtain the dually transformed Hamiltonian:

$$H'_\infty = \sum_x \left[\frac{1}{2\beta J} \mathbf{b}_x^2 + i\mathbf{b}_x \cdot \mathbf{A}_x + i\frac{\theta}{2} \mathbf{A}_x \cdot (\boldsymbol{\nabla} \times \mathbf{A}_x) \right], \tag{6}$$

where the integer variables $\mathbf{b}_x$ satisfy the constraint $\boldsymbol{\nabla} \cdot \mathbf{b}_x = 0$ arising from the φ_x integration, meaning that only configurations with closed vortex loops are counted [16]. We introduce a second integer variable $\tilde{\mathbf{a}}_x$ such that $\mathbf{b}_x = \boldsymbol{\nabla} \times \tilde{\mathbf{a}}_x$. Using the Poisson formula to convert the sum over $\tilde{\mathbf{a}}_x$ to an integral over $\tilde{\mathbf{A}}_x$ and an auxiliary sum over integers $\tilde{\mathbf{b}}_x$ yields

$$\begin{aligned} H''_\infty &= \sum_x \Bigg[\frac{1}{2\beta J} (\boldsymbol{\nabla} \times \tilde{\mathbf{A}}_x)^2 + i\mathbf{A}_x \cdot (\boldsymbol{\nabla} \times \tilde{\mathbf{A}}_x) \\ &+ i\frac{\theta}{2} \mathbf{A}_x \cdot (\boldsymbol{\nabla} \times \mathbf{A}_x) + 2\pi i \tilde{\mathbf{b}}_x \cdot \tilde{\mathbf{A}}_x \Bigg], \end{aligned} \tag{7}$$

where $\boldsymbol{\nabla} \cdot \tilde{\mathbf{b}}_x = 0$. Integrating out $\mathbf{A}_x$, we obtain

$$\begin{aligned} \tilde{H}_\infty &= \sum_x \Bigg[\frac{1}{2\beta J} (\boldsymbol{\nabla} \times \tilde{\mathbf{A}}_x)^2 + i\frac{1}{2\theta} \tilde{\mathbf{A}}_x \cdot (\boldsymbol{\nabla} \times \tilde{\mathbf{A}}_x) \\ &\quad + i\, 2\pi \tilde{\mathbf{b}}_x \cdot \tilde{\mathbf{A}}_x + \frac{\epsilon_0}{2} \tilde{\mathbf{b}}_x^2 \Bigg], \end{aligned} \tag{8}$$

where we have added an extra core energy to the $\tilde{\mathbf{b}}_x$ field, thereby generalizing slightly the model which has $\epsilon_0 = 0$. The core energy allows us to introduce an auxiliary field $\tilde{\varphi}_x$ and rewrite the partition function as

$$\begin{aligned} H_\infty^{\text{dual}} &= \sum_x \Bigg[\sum_\mu \frac{1}{2\epsilon_0} \left(\boldsymbol{\nabla}_\mu \tilde{\varphi}_x - 2\pi \tilde{\mathbf{n}}_x - \tilde{\mathbf{A}}_x \right)^2 \\ &+ \frac{1}{8\pi^2 \beta J} (\boldsymbol{\nabla} \times \tilde{\mathbf{A}}_x)^2 + i\frac{1}{8\pi^2 \theta} \tilde{\mathbf{A}}_x \cdot (\boldsymbol{\nabla} \times \tilde{\mathbf{A}}_x) \Bigg], \end{aligned} \tag{9}$$

where we have rescaled $\tilde{\mathbf{A}}_x \to \tilde{\mathbf{A}}_x/2\pi$. The first term arises from $\epsilon_0 \tilde{\mathbf{b}}_x^2/2 + i\, 2\pi \tilde{\mathbf{b}}_x \cdot \tilde{\mathbf{A}}_x$ in Eq. (8) by the same standard technique described before Eq. (6) for the same untilded quantities, although in the opposite direction. The dual Hamiltonian (9) has precisely the same form as (4), and coincides with it if we replace $\epsilon_0 \to (\beta J)^{-1}$, $e^2 \to 4\pi^2 \beta J$, $\theta \to 1/(4\pi^2 \theta)$. The dual Hamiltonian is the limit $\epsilon_0 \to 0$ of (8), in which case we speak with Peskin [17] of a "frozen" limit of the GLCS model. Thus, up to smooth factors in the temperature, the free energy satisfy the duality relation:

$$F(T, \theta, e = \infty) = F\left(T' = 0,\ \theta' = \frac{1}{4\pi^2 \theta},\ e' = 2\pi \sqrt{\frac{J}{T}} \right). \tag{10}$$

From Eq. (10) we see that the fixed point of the duality transformation is given by

$$T = 0, \qquad \theta' = \theta = \frac{1}{2\pi}. \tag{11}$$

At this point, the Hamiltonian H_∞ is self-dual. Note that in the absence of the CS-term, the dual of the frozen superconductor is merely a Villain model as observed by Peskin [17], and there is no self-duality.

The Villain model can well be approximated by a XY-model which, in turn, can be transformed into a complex disorder field theory [16]. The result is

$$\begin{aligned} \mathcal{H}_\infty^{\text{dual}} &= \frac{1}{8\pi^2 g^2} (\boldsymbol{\nabla} \times \tilde{\mathbf{A}})^2 + i\frac{1}{8\pi^2 \theta} \tilde{\mathbf{A}} \cdot (\boldsymbol{\nabla} \times \tilde{\mathbf{A}}) \\ &+ \left| \left(\boldsymbol{\nabla} - i\tilde{\mathbf{A}} \right) \phi \right|^2 + r' |\phi|^2 + \frac{u'}{2} |\phi|^4, \end{aligned} \tag{12}$$

where $g^2 \equiv \beta J \Lambda$, and Λ is an ultraviolet cutoff reminiscent of the lattice. Interestingly, the local gauge invariance of the disordered phase of $\mathcal{H}_\infty$ is also present in the

disordered phase of $\mathcal{H}_\infty^{\rm dual}$. This does not happen in the GL model, where the disordered phase of the corresponding dual theory has no local gauge symmetry [15, 16, 19]. Note that if we set $\theta = 1/2\pi$ in Eq. (12), we obtain a continuum dual Hamiltonian similar to the one considered in Ref. 13, except that the coefficient of the Maxwell term is not equal to unity. The continuum dual Hamiltonian (12) was obtained for the first time in the context of the fractional quantum Hall effect by Wen and Niu [20]. Their derivation, however, was only heuristic and performed in the continuum, in contrast to our derivation by an *exact* duality transformation on the lattice.

Near the critical region, where the lattice model has a good continuum limit, the Maxwell term is irrelevant with respect to the CS-term [11], which tells us that the continuum dual model (12) becomes self-dual for $\theta = 1/2\pi$. *This observation implies that an interacting fermion model in three dimensions corresponds to the fixed point of a duality transformation in the* GLCS model. This self-dual point marks a phase transition in the GLCS model as the statistical parameter θ is varied. This phase transition is associated with the specialization of anyonic statistics to fermionic statistics.

In the original Hamiltonian $\mathcal{H}_\infty$, the β-function $\beta_{\theta_r} \equiv \mu\partial\theta_r/\partial\mu = 0$ [11, 21], with θ_r being the renormalized statistical parameter. The β-function vanishes because the anomalous dimension of the the vector potential vanishes, $\eta_A = 0$. This is in contrast to the pure GL model, where $\eta_A = 1$ [22]. In our model, the scaling dimension of $\mathbf{A}$ is equal to the canonical one, $[\mathbf{A}] = 1$. This makes the scaling dimension of the Maxwell term equal to four, and thus irrelevant in the long-wavelength limit.

In the dual model, it is the β-function of the ratio $\hat{g}_r^2/\theta_r$ that vanishes at the critical point. Here, $\hat{g}_r^2 = g_r^2/\mu$ is the renormalized dimensionless gauge coupling of the disorder field theory. Thus, the β-functions of $\hat{g}_r^2$ and θ_r in the disorder field theory are given by

$$\beta_{\hat{g}^2} = (\eta_{\tilde{A}} - 1)\hat{g}_r^2, \qquad \beta_{\theta_r} = (\eta_{\tilde{A}} - 1)\theta_r, \tag{13}$$

where $\eta_{\tilde{A}} \equiv \mu\partial\ln Z_{\tilde{A}}/\partial\mu$, with $Z_{\tilde{A}}$ being the wave function renormalization of the dual gauge field. From (13) we see that a charged fixed point corresponds to $\eta_{\tilde{A}} = 1$, in which case θ_r can assume any value. Due to the presence of the Maxwell term, the canonical dimension of $\tilde{\mathbf{A}}$ is two and since $\eta_{\tilde{A}} = 1$, the scaling dimension of $\tilde{\mathbf{A}}$ is unity. Thus, we see that $\mathbf{A}$ and its dual $\tilde{\mathbf{A}}$ have the same scaling dimension, that is, $[\mathbf{A}] = [\tilde{\mathbf{A}}] = 1$, and we can speak of a self-duality of the gauge field scaling dimension. In the GL model, the scaling dimensions behave differently under duality. Indeed, in the GL model, we have $\eta_A = 1$ [22], whereas the corresponding disorder field theory has $\eta_{\tilde{A}} = 0$ [19, 23, 24] and, since the Maxwell term is present in both theories, we have $[\mathbf{A}] = 1$ and $[\tilde{\mathbf{A}}] = 2$.

Summarizing, we have found a nontrivial self-dual model in three dimensions as a fixed point in an exact duality transformation of a lattice model containing a scalar model minimally coupled to a gauge field, whose dynamics is governed by a CS term. The lattice dual model contains a Maxwell term that vanishes at the self-dual point. This lies at the anyonic statistics parameter $\theta_c = 1/2\pi$, corresponding to Fermi statistics. The self-duality implies that there exists a phase transition if the statistics parameter is varied. We have also derived the continuum limit of the dual model as a disorder field theory. It was shown that the original gauge field and its dual have the same scaling dimension. An interesting topic to study in the future concerns the order of the statistics-parameter-induced phase transition.

The work of FSN is supported by the Alexander von Humboldt foundation.

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